2) I throw a fair die repeatedly. Let $Y_0, Y_1, Y_2, \cdots Y_n, \cdots$ be the numbers rolled (they are i.i.d. uniform on $\{1,2,3,4,5,6\}$). We define $(X_n)_{n\geq 0}$ by $X_n = \max\{Y_0, Y_1, Y_2, \cdots Y_n\}$ (so with probability one the chain is absorbed at state 6).
(i) $(X_n)_{n\geq 0}$ is a (λ, P) Markov chain. (You need not prove it is a Markov chain!). What is λ and P?
(ii) Is the chain aperiodic? Irreducible?
(iii) For $i \in \{1,2,3,4,5,6\}$, let function $h(i) = P(T_4 < \infty | X_0 = i)$. Find $h(i)$ for each i where as before. $T_4 = \inf\{n \geq 0 : X_n = 4\}$. Give corresponding linear equations for h. Is $P(T_4 < \infty | X_0 = i)$, the only solution for these equations?
(iv) Calculate the expected time the process spends in state 5.

(i) $X_0 = Y_0$ WHICH IS OF DISN UNIFORM ON $\{1,2..6\}$

~~THE~~ SO $\lambda = (\frac{1}{6}, \frac{1}{6} \cdots \frac{1}{6})$

$P_{ij} = 0$ FOR $j < i$ $\quad P_{ii} = \frac{i}{6}$ AS THIS IS

PROB $Y_{n+1} \leq i \; \forall n$ $\quad P_{ij} = \frac{1}{6} \; \forall j > i$

(ii) SINCE $P_{ii} > 0 \; \forall i \in I$ EACH POINT HAS PERIOD 1

SO CHAIN IS APERIODIC; THE CHAIN IS NOT IRREDUCIBLE AS

6 IS ABSORBING $P_{6i} = 0 \; \forall i < 6$, MORE GENERALLY

$i \not\to j$ FOR $j < i$

(iii) $h(5) = h(6) = 0 \quad h(4) = 1$

FOR $i < 4$ $\quad h(i) = \frac{i}{6} h(i) + \frac{1}{6} h(i+1) + \frac{1}{6} h(i+1) \cdots + \frac{1}{6} h(6)$

$= \frac{i}{6} h(i) + \frac{1}{6} h(i+1) \cdots + \frac{1}{6} h(3) + \frac{1}{6}$.

$\Rightarrow h(3) = \frac{1}{2} h(3) + \frac{1}{6} h(4) = \frac{1}{2} h(3) + \frac{1}{6} \Rightarrow \underline{h(3) = 1/3}$

$\Rightarrow h(2) = \frac{1}{3} h(2) + \frac{1}{6} h(3) + \frac{1}{6} h(4) = \frac{1}{3} h(2) + \frac{1}{18} + \frac{1}{6}$

$\underline{h(2) = 1/3}$

$h(1) = \frac{1}{6} h(1) + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \qquad \underline{h(1) = 1/3}$

(iii) CONTD

YES $h(3)$ IS UNIQUELY DEFINED

SO RECURSIVELY $h(2)$ AND THEN $h(1)$ ARE UNIQUELY DEFINED.

(iv) THE PROBABILITY THAT $T_5 < \infty$ IS $1/2$

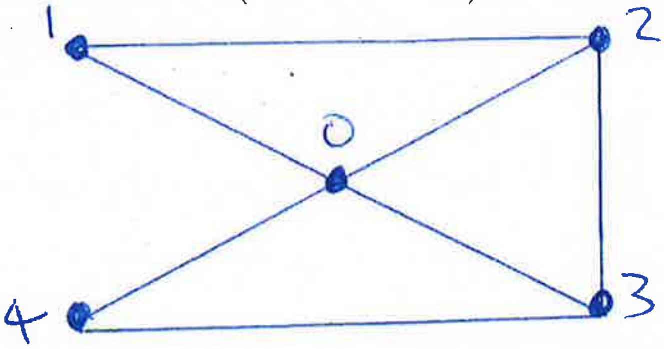
INDEED $\forall\ Y_0 < 5$ WE CAN SAY $T_5 < \infty \iff$

A 5 IS ROLLED BEFORE A 6. WE COULD OTHERWISE REDO (iii) REPLACING 4 WITH 5.

GIVEN $T_5 < \infty$ THE # OF VISITS TO 5 IS A GEOMETRIC$\left(\frac{1}{6}\right)$ SO ITS EXPECTATION IS $\frac{1}{1/6} = 6$

SO EXPECTED # OF VISITS TO 5 $= \frac{1}{2} 6 = \underline{\underline{3}}$

3)What does it mean for a Markov chain $(X_n)_{n\geq 0}$, with transition matrix P, to satisfy the detailed balence equations for measure π on the state space?
(ii)For an irreducible continuous time chain $(X_t)_{t\geq 0}$, with $Q-$matrix Q, satisfying the detailed balance equations with measure π on the state space, show that the measure π is invariant for Q.
(iii) Give a criterion for X, as in (ii), to be positive recurrent (given that π is a distribution).
(iv) Suppose now that X is a (continuous time) random walk on the graph



so for the jump chain P_{ij} is zero if i and j are not neighbors and is otherwise $\frac{1}{d_i}$ where d_i is the degree of vertex i on the graph. The jump rate $q_i = 1$ for each state i. Find the equilibrium π. Can you generalize to general finite connected graphs satisfying these conditions? Is it true that for any X_0, $P(X_t = i) \to \pi(i)$? (Recall that a graph is connected if there exists a path between any two distinct sites.)

(i) $(X_n)_{n\geq 0}$ SATISFIES DETAILED BALANCE WRT π IF $\forall\, i,j$

$$\pi_i P_{ij} = \pi_j P_{ji}$$

(ii) FOR CTS TIME MC $(X_t)_{t\geq 0}$ IT SATISFIES DB WRT MEASURE π IF $\forall\, i \neq j$ $\quad \pi_i q_{ij} = \pi_j q_{ji}$

IF SO THEN $(\pi Q)_i = \sum_j \pi_j q_{ji} = -\pi_i q_i + \sum_{j\neq i} \pi_j q_{ji}$

$\underline{DB}$ $\quad -\pi_i q_i + \sum_{j\neq i} \pi_i q_{ij} = \pi_i \left(\sum_{j\neq i} q_{ij} - q_i \right) = 0.$

(iii) THE CHAIN IS NON EXPLOSIVE

(iv) $\forall\, i \neq j$ $\quad \pi_i q_{ij}$ (FOR $\pi_i = d_i$) $= 1$ IF $i \sim j$
$= 0$ IF $i \not\sim j$.

SO IN EITHER CASE $\pi_i q_{ij} = \pi_j q_{ji}$ SO EQUILIBRIUM (UNIQUE AS STATE SPACE FINITE & CHAIN IRREDUCIBLE)
IS $\quad \pi_i = \dfrac{d_i}{\sum_j d_j}$. $\quad$ YES $P(X_t = i) \to \pi(i)$
AS CHAIN IS IN CTS TIME

(V) BY SIMILAR CALCULATIONS

$$P(T_5 < \infty) = 1/2 \text{ AND}$$

GIVEN $T_5 < \infty$ THE # OF TIMES $X_n = 5$ IS A GEOMETRIC$\left(\frac{1}{6}\right)$

(i.e WITH EXPECTATION $1/{1/6} = 6$)

SO THE EXPECTED TIME SPENT IN 5 IS $\frac{1}{2}\, 6 = 3$